Definition 1. *Let $\mathscr{N}$ denote the set of all measures with natural spectra, i. e.*

$$\mathscr{N} = \{\mu \in M(\mathbb{T}) : \overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)\}.$$

It is proved in [Z] that this set is not closed under addition. In the following proposition we obtain few properties of $\mathscr{N}$ which will be used later.

Proposition 2. *The set of all measures with natural spectra is involutive ($\mu \in \mathscr{N} \Rightarrow \widetilde{\mu} \in \mathscr{N}$), closed subset of $M(\mathbb{T})$ which is also closed under multiplication by complex numbers. Moreover, this set is closed under an action of the functional calculus i. e. for $\mu \in \mathscr{N}$ and a holomorphic function defined on some open neighborhood of $\sigma(\mu)$ we have $f(\mu) \in \mathscr{N}$.*

Proof. Let us take $\mu \in \mathscr{N}$. Then it is obvious that $\alpha\mu \in \mathscr{N}$ for all $\alpha \in \mathbb{C}$. Moreover, $\sigma(\widetilde{\mu}) = \overline{\sigma(\mu)}$ and $\widehat{\widetilde{\mu}}(n) = \overline{\widehat{\mu}}(n)$ for $n \in \mathbb{Z}$ shows involutivness of $\mathscr{N}$. Let $(\mu_n)_{n=1}^{\infty} \subset \mathscr{N}$ satisfy $||\mu_n - \mu|| \to 0$ as $n \to \infty$ for some $\mu \in M(\mathbb{T})$ and fix $\varepsilon > 0$. Then for any $\varphi \in \mathfrak{M}(M(\mathbb{T}))$ and $k \in \mathbb{Z}$ we have

$$|\widehat{\mu}(\varphi) - \widehat{\mu}(k)| \leq |\widehat{\mu}(\varphi) - \widehat{\mu_n}(\varphi)| + |\widehat{\mu_n}(\varphi) - \widehat{\mu_n}(k)| + |\widehat{\mu_n}(k) - \widehat{\mu}(k)| \leq$$
$$2||\mu - \mu_n|| + |\widehat{\mu_n}(\varphi) - \widehat{\mu_n}(k)|.$$

Now, we fix $n_0 \in \mathbb{N}$ such that $||\mu - \mu_{n_0}|| < \frac{\varepsilon}{3}$. Since $\mu_{n_0} \in \mathscr{N}$ there exists $k_{n_0} \in \mathbb{N}$ satisfying $|\widehat{\mu_{n_0}}(\varphi) - \widehat{\mu_{n_0}}(k_{n_0})| < \frac{\varepsilon}{3}$ and we get

$$|\widehat{\mu}(\varphi) - \widehat{\mu}(k_{n_0})| < \varepsilon.$$

Since the spectrum of an element in a commutative Banach algebra is an image of its Gelfand transform, the first part of the proof is finished.
Let us take $\mu \in \mathscr{N}$ and f - a holomorphic function defined on some open neighborhood of $\sigma(\mu)$. Then f acts on μ and so there exists $\nu := f(\mu)$ such that

$$\underset{\varphi \in \mathfrak{M}(M(\mathbb{T}))}{\forall} \varphi(\nu) = f(\varphi(\mu)). \tag{1}$$

Let $\lambda \in \sigma(\nu)$. Then, from the spectral mapping theorem ($\sigma(\nu) = f(\sigma(\mu))$) we can find $\alpha \in \sigma(\mu)$ satisfying $f(\alpha) = \lambda$. Since $\mu \in \mathscr{N}$ there is a sequence (n_k) for which

$$\lim_{k\to\infty} \widehat{\mu}(n_k) = \alpha.$$

Using continuity of f and (1) we have

$$\lambda = f(\alpha) = \lim_{k\to\infty} f(\widehat{\mu}(n_k)) = \lim_{k\to\infty} \widehat{\nu}(n_k).$$

Hence $\sigma(\nu) = \overline{\widehat{\nu}(\mathbb{Z})}$ and the whole proof is finished. □

The set $\mathscr{N}$ does not have Banach algebra structure so it is convenient to introduce the set of 'suitable perturbations'.

Definition 3. *We say that a measure $\mu \in M(\mathbb{T})$ is* ***spectrally reasonable****, if $\mu + \nu \in \mathscr{N}$ for all $\nu \in \mathscr{N}$. The set of all spectrally reasonable measures will be denoted by $\mathscr{S}$.*

It is clear from the definition that the set $\mathscr{S} \subset \mathscr{N}$ is closed under addition and multiplication by complex numbers. Before we show that it has Banach algebra structure we will prove an auxiliary lemma.

Lemma 4. *Spectrally reasonable measures have the following properties:*

1. *If $\mu \in \mathscr{S}$ and $\nu \in \mathscr{N}$, then $\mu * \nu \in \mathscr{N}$.*
2. *If $\mu \in \mathscr{S}$ is invertible, then $\mu^{-1} \in \mathscr{S}$.*

Proof. Let us take $\mu \in \mathscr{S}$ and consider first only invertible $\nu \in \mathscr{N}$. Then, by the spectral mapping theorem $\nu^{-1} \in \mathscr{N}$. From the definition of $\mathscr{S}$ we have $\mu + \nu^{-1} \in \mathscr{N}$. Now,

$$\lambda \in \sigma(\mu * \nu) \Leftrightarrow 0 \in \sigma(\mu * \nu - \lambda\delta_0) \Leftrightarrow 0 \in \sigma(\mu - \lambda\nu^{-1}).$$

Since the set $\mathscr{N}$ is closed under multiplication by scalars $-\lambda\nu^{-1} \in \mathscr{N}$ which by the definition of $\mathscr{S}$ leads to $\mu - \lambda\nu^{-1} \in \mathscr{N}$. Hence, there exists a sequence of integers (n_k) such that

$$\lim_{k\to\infty} \left(\widehat{\mu}(n_k) - \frac{\lambda}{\widehat{\nu}(n_k)} \right) = 0.$$

This is obviously equivalent to

$$\lim_{k\to\infty} \widehat{(\mu * \nu)}(n_k) = \lambda$$

which shows that $\mu * \nu \in \mathscr{N}$. For general $\nu \in \mathscr{N}$ we take $\alpha \in \mathbb{R}_+$ such that $\nu + \alpha\delta_0$ is invertible (we may put any $\alpha > r(\nu)$). Then

$$\mu * \nu = \mu * (\nu + \alpha\delta_0 - \alpha\delta_0) = \mu * (\nu + \alpha\delta) - \alpha\mu.$$

From the earlier part of the proof $\mu * (\nu + \alpha\delta) \in \mathscr{N}$ and finally $\mu * \nu \in \mathscr{N}$ which gives the first claim of the lemma.

We move to the second statement. Let us take an invertible $\mu \in \mathscr{S}$ and $\nu \in \mathscr{N}$. Then, similarly to previous arguments we have

$$\lambda \in \sigma(\mu^{-1} + \nu) \Leftrightarrow 0 \in \sigma(\mu^{-1} + \nu - \lambda\delta_0) \Leftrightarrow 0 \in \sigma(\mu * \nu + \delta_0 - \lambda\mu).$$

From the first part of the lemma we have $\mu * \nu + \delta_0 \in \mathscr{N}$ which gives the desired conclusion in exactly the same way as before. □

We are ready now to show that $\mathscr{S}$ has Banach algebra structure.

Theorem 5. *The set $\mathscr{S}$ is closed, unital *-subalgebra of $M(\mathbb{T})$.*

Proof. Closedness and involutivness of $\mathscr{S}$ follows directly from Proposition 2. Of course, $\delta_0 \in \mathscr{S}$ and so it is enough to prove that if $\mu_1, \mu_2 \in \mathscr{S}$, then $\mu_1 * \mu_2 \in \mathscr{S}$. Let us take $\nu \in \mathscr{N}$ and assume first that μ_2 is invertible. Then

$$\lambda \in \sigma(\mu_1 * \mu_2 + \nu) \Leftrightarrow 0 \in \sigma(\mu_1 * \mu_2 + \nu - \lambda\delta_0) \Leftrightarrow 0 \in \sigma(\mu_1 - \lambda\mu_2^{-1} + \nu * \mu_2^{-1}).$$

From the previous lemma (second part) $\mu_2^{-1} \in \mathscr{S}$ and so $\mu_1 - \lambda\mu_2^{-1} \in \mathscr{S}$. Moreover, from the the first part of the last lemma $\nu * \mu_2^{-1} \in \mathscr{N}$ which shows $\mu_1 - \lambda\mu_2^{-1} + \nu * \mu_2^{-1} \in \mathscr{N}$ and we are able to proceed analogously as in the proof of the lemma. For general μ_2 we take once again $\alpha \in \mathbb{R}_+$ such that $\mu_2 + \alpha_{\delta_0}$ is invertible and then

$$\mu_1 * \mu_2 + \nu = \mu_1 * (\mu_2 + \alpha\delta_0) - \alpha\mu_1 + \nu.$$

From the first part we obtain $\mu_1 * (\mu_2 + \alpha\delta_0) - \alpha\mu_1 \in \mathscr{S}$ which finishes the proof. □

Now, we will examine other relevant features of $\mathscr{S}$.

Proposition 6. *The algebra $\mathscr{S}$ is a symmetric Banach *-algebra, i.e.*

$$\forall_{\varphi \in \mathfrak{M}(\mathscr{S})} \forall_{\mu \in \mathscr{S}} \varphi(\widetilde{\mu}) = \overline{\varphi(\mu)}.$$

Proof. This is not difficult, since for $\mu = \widetilde{\mu} \in \mathscr{S}$ we know that $\sigma(\mu) \subset \mathbb{R}$ which gives

$$\varphi(\widetilde{\mu}) = \varphi(\mu) = \overline{\varphi(\mu)} \text{ for all } \varphi \in \mathscr{S}.$$

For general $\mu \in \mathscr{S}$ we use standard decomposition into hermitian and anti-hermitian part

$$\mu = \frac{\mu + \widetilde{\mu}}{2} + i\frac{\mu - \widetilde{\mu}}{2i}$$

and the result follows from the previous argument. □

The last proposition leads to the corollary which sheds some light on the structure of $\mathfrak{M}(\mathscr{S})$.

Theorem 7. *The set $\mathbb{Z}$ identified with functionals $\mu \mapsto \widehat{\mu}(n)$ is dense in $\mathscr{S}$.*

Proof. Let $\widehat{\mathscr{S}} = \{\widehat{\mu} : \mu \in \mathscr{S}\}$. The assertion of Proposition 6 implies that $\widehat{A} \subset C(\mathfrak{M}(\mathscr{S}))$ is a self-adjoint subalgebra which contains constant function. Hence from the Stone - Weierstrass theorem $\widehat{A}$ is dense in $C(\mathfrak{M}(\mathscr{S}))$.
Let us assume on the contrary that $\overline{\mathbb{Z}} \neq \mathfrak{M}(\mathscr{S})$. Then from the Urysohn's lemma (in fact it follows just from complete regularity of $\mathfrak{M}(\mathscr{S})$) there exists $f \in C(\mathfrak{M}(\mathscr{S}))$ such that $f|_{\overline{\mathbb{Z}}} \equiv 0$ and $f(\varphi) = 1$ for some $\varphi \in \mathfrak{M}(\mathscr{S}) \setminus \overline{\mathbb{Z}}$. Using the density of $\widehat{A}$ in $C(\mathfrak{M}(\mathscr{S}))$ we find $\mu \in \mathscr{S}$ such that $||\widehat{\mu} - f||_{C(\mathfrak{M}(\mathscr{S}))} < \frac{1}{3}$. It gives $|\widehat{\mu}|_{\mathbb{Z}} < \frac{1}{3}$ and $|\widehat{\mu}(\varphi)| > \frac{2}{3}$. This is impossible because $\mu \in \mathscr{N}$ and so there exists a sequence (n_k) satisfying

$$\lim_{k \to \infty} \widehat{\mu}(n_k) = \widehat{\mu}(\varphi).$$

□

We will determine some members of $\mathscr{S}$. In our paper we have proved that the sum of two measures with natural spectra has natural spectrum if one of summands has Fourier coefficients tending to zero. The set of measures in $M_0(\mathbb{T})$ with natural spectrum is exactly the Zafran's ideal $\mathscr{C}$. Thus, in our present terminology we can formulate this result as follows.

Theorem 8. *$\mathscr{C} \subset \mathscr{S}$.*

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