Twisted sums with $C(K)$-spaces

(Working group in applications of set theory)

Abstract: Based on the paper by Cabello Sánchez, Castillo, Kalton and Yost published in Trans. Amer. Math. Soc. 355(11) 2003, we present several results on the (non-) existence of non-trivial twisted sums $0 \to C(K) \to Y \to X \to 0$, where K is either $[0,1]$ or $[0,\omega^\omega]$ and X, Y are Banach spaces. In particular, we are interested in characterizing those spaces X for which there exists a twisted sum as above with a strictly singular quotient map. For $K = [0,1]$ and separable X, we prove that such a twisted sum exists if and only if X contains no copy of ℓ_1; this leads to a construction of a twisted sum of $C[0,1]$ and c_0 (which is thus necessarily an $\mathscr{L}_\infty$-space) that is not isomorphic to any quotient of a $C(K)$-space. For $K = [0,\omega^\omega]$, we show that such a twisted sum with a strictly singular quotient map exists, provided that X admits an unconditional finite-dimensional Schauder decomposition and contains no subspace isomorphic to the dual of a Banach space with summable Szlenk index. This leads to a construction of a 'Bourgain–Delbaen type' space, namely, an $\mathscr{L}_\infty$-space which is a predual of ℓ_1, yet is not isomorphic to any quotient of a $C(K)$-space.